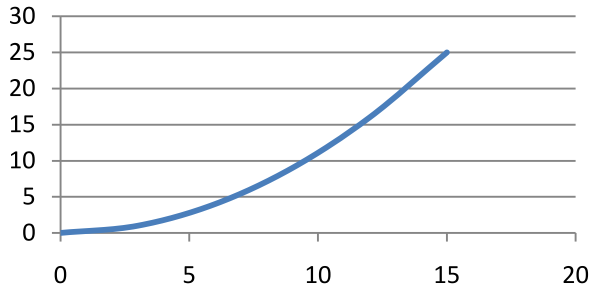
Parametric Differentiation

Parametric Equations in x and y are ones where both x and y are defined in terms of a third variable, called a **parameter**. The parameter is often t, representing time.

Example: $\begin{Bmatrix} x = 3t \\ y = t^2 \end{Bmatrix}$ are the parametric equations for an object that starts at the origin at time 0, and its position relative to the x-axis after t seconds is given by $x = 3t$ and its position relative to the y-axis after t seconds is given by $y = t^2$.

When time, t, is known, the (x, y) position can be calculated:

t	0	1	2	3	4	5
$x = 3t$	0	3	6	9	12	15
$y = t^2$	0	1	4	9	16	25



We use the Chain Rule to differentiate parametric equations:

Chain Rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

depending on whether the 3rd parameter is t or θ.

Example 1: Differentiate the following parametric equation: $x = 6t^2, \ y = 2t$.

$\frac{dy}{dt} = 2$ $\qquad \frac{dx}{dt} = 12t$ $\qquad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2 \times \frac{1}{12t} = \frac{2}{12t} = \frac{1}{6t}$

Example 2: The following parametric equation describes a circle: $x = 2\cos\theta, \ y = 2\sin\theta$. Differentiate to find $\frac{dy}{dx}$.

$\frac{dy}{d\theta} = 2\cos\theta$ $\qquad \frac{dx}{d\theta} = -2\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2\cos\theta \times \frac{1}{-2\sin\theta} = \frac{2\cos\theta}{-2\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$$

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Equations of tangents and normals

Example: The parametric equations $\begin{Bmatrix} x = (t+1)^3 \\ y = t^2 - 2 \end{Bmatrix}$ define a curve. Calculate the equations of the tangent and the normal to the curve at the point (8, -1).

$$\frac{dy}{dt} = 2t \qquad \frac{dx}{dt} = 3 \cdot (t+1)^{3-1} \cdot 1 = 3(t+1)^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{3(t+1)^2} = \frac{2t}{3(t+1)^2}$$

We need a 't' value for the coordinate (8, -1):

When $x = 8$, $(t+1)^3 = 8 \rightarrow t+1 = \sqrt[3]{8} \rightarrow t+1 = 2 \rightarrow t = 1$

When $y = -1, t^2 - 2 = -1 \rightarrow t^2 = 1 \rightarrow t = \pm 1 \rightarrow t = 1$

$$m(1) = \frac{2(1)}{3(1+1)^2} = \frac{2}{12} = \frac{1}{6} \qquad m_n = \frac{-1}{1/6} = -6$$

Eqn of tangent at point (8, -1): $y - (-1) = \frac{1}{6}(x-8)$

$$y + 1 = \frac{1}{6}x - 1\frac{1}{3}$$

$$y = \frac{1}{6}x - 2\frac{1}{3}$$

Eqn of normal at point (8, -1): $y - (-1) = -6(x-8)$

$$y + 1 = -6x + 48$$

$$y = -6x + 47$$

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